# An Algorithm for Fitting 2-D Data on the Circle: Applications to Mobile Robotics

P. Núñez, R. Vázquez-Martín, A. Bandera, and F. Sandoval, *Member, IEEE*

***Abstract*—In this paper, an approach for fitting a circle to 2-D data which represent only a small part of the curve is described. This approach deals with the particular case where data is specified in terms of its Cartesian coordinates and the errors in both coordinates are not independent. Besides, an associated uncertainty ellipse which describes the measurement error and the variance matrix associated to the estimated parameters are obtained. This method is particularly well designed to fit a circle to a set of measured range readings provided by a 2-D laser range finder when these range readings are specified in terms of its Cartesian coordinates. Therefore, it has been successfully applied to acquire circle-shaped landmarks in a mobile robotics navigation task.**

***Index Terms*—Circle fitting, laser landmark acquisition, mobile robotics.**

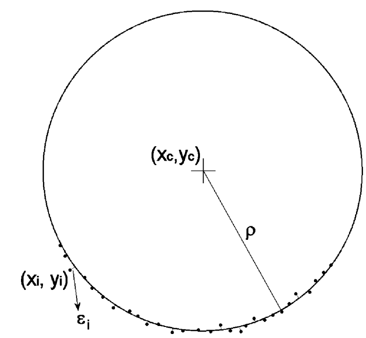


Fig. 1. Problem statement: $(x_c, y_c)$ and $\rho$ denote the centre coordinates and radius of the circle $C$ which yields the best fit to the measured data points $\{x_i, y_i\}_{|i=1...m}$.

## I. Introduction

FITTING circles to bidimensional data is one of the basic problems in pattern recognition, computer vision, metrology or microwave measurement [1]. Although many circle fitting methods have been proposed, it is a common choice to achieve this by minimizing the mean square distance from the data points to the fitting circle. Basically, the least squares fit (LSF) assumes that each data point is the noised version of the model point which is closest to it. This assumption is valid when data points are not contaminated by strong noise.

In a recent study, Chernov and Lesort [5] conclude that conventional fitting algorithms which are based on the LSF approach become unstable when applied to incomplete data, specially when data are sampled along a small arc. To solve this problem, they combine two existing algorithms, building a new approach which remains stable when applied to short arcs. This approach selects an initial guess using a gradient weighted algebraic fit. Then, it uses an iterative scheme to minimize the mean square distance from the fitting circle to data points. In this paper, we extend this previous work to deal with the case in which the errors in both coordinates of the range readings are not independent. Besides, the described method is able to represent uncertainties and to propagate them from single points to all stages involved in the circle parameter estimation process. Thus, in addition to estimate the circle parameters, this method provides their associated variance matrix.

Manuscript received July 25, 2007; revised September 26, 2007. This work was supported in part by a grant from the Spanish Ministerio de Educación y Ciencia (MEC) under Project TIN2005-01359. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Dominic Ho.

P. Núñez is with Department Tecnología de los Computadores y las Comunicaciones, Universidad de Extremadura, Spain (e-mail: pmnt@uma.es).

R. Vázquez-Martín, A. Bandera, and F. Sandoval are with the Grupo de Ingeniería de Sistemas Integrados, Department Tecnología Electrónica, Universidad de Málaga, 29071-Málaga, Spain (e-mail: rvmartin@uma.es; ajbandera@uma.es, sandoval@dte.uma.es).

Digital Object Identifier 10.1109/LSP.2007.912964

## II. Problem Statement

Let the data points be $\{x_i, y_i\}_{|i=1...m}$ $(m > 3)$, with an uncertainty ellipse specified in terms of the standard deviations $p_i$ and $q_i$, and the correlations $r_i$. The problem is to obtain the centre $(x_c, y_c)$ and the radius $\rho$ of the circle $C$ which yields the best fit to this data. It is also required to determine the variance matrix associated to the circle parameters.

This problem is stated as the minimization of the difference between the set of points $\{x_i, y_i\}$ and their corresponding points $\{x_c + \rho\cos\varphi_i, y_c + \rho\sin\varphi_i\}$ which lie on $C$ (Fig. 1). This difference is summarized by the $2m$-element error vector $\varepsilon$

$$\begin{aligned}\varepsilon &= (x_1 - (x_c + \rho\cos\varphi_1), y_1 - (y_c + \rho\sin\varphi_1), \dots,\\ &\quad x_m - (x_c + \rho\cos\varphi_m), y_m - (y_c + \rho\sin\varphi_m))^T\\ &= (\Delta x_1, \Delta y_1, \dots \Delta x_m, \Delta y_m)^T. \end{aligned} \tag{1}$$

This error vector has the known $2m \times 2m$ block diagonal variance matrix $V = diag(V_1 \dots V_m)$, where

$$V_i = \begin{bmatrix} p_i^2 & r_i \\ r_i & q_i^2 \end{bmatrix}. \tag{2}$$

Then, assuming that the errors are normally distributed, the *maximum likelihood* (ML) problem consists of minimizing

$$\text{minimize} \quad \varepsilon^T V^{-1} \varepsilon \tag{3}$$

with respect to the vector $b = (\varphi_1, \dots, \varphi_m, x_c, y_c, \rho)^T$.

In order to convert the problem stated in (3) into an unweighted one, the variance matrix $V$ can be factorized such that $V = LL^T$, where $L$ is lower triangular [4]. In this case

$$\begin{aligned}\varepsilon^T V^{-1}\varepsilon &= \varepsilon^T (LL^T)^{-1}\varepsilon \\ &= (\varepsilon^T L^{-T})(L^{-1}\varepsilon) = (L^{-1}\varepsilon)^T L^{-1}\varepsilon = \varepsilon'^T\varepsilon' \end{aligned} \quad (4)$$

and the problem has been reduced to the minimization of

$$\varepsilon'^T\varepsilon' = \sum_{i=1}^{m}(\Delta x_i'^2 + \Delta y_i'^2) \quad (5)$$

being the vector $\varepsilon_i' = (\Delta x_i', \Delta y_i')^T$ defined by

$$\varepsilon_i' = L_i^{-1}\begin{bmatrix}\Delta x_i \\ \Delta y_i\end{bmatrix}. \quad (6)$$

This vector is equal to

$$\varepsilon_i' = \begin{bmatrix} p_i^{-1} & 0 \\ \frac{-r_i}{p_i^2 q_i\sqrt{1-\left(\frac{r_i}{p_i q_i}\right)^2}} & \frac{1}{q_i\sqrt{1-\left(\frac{r_i}{p_i q_i}\right)^2}}\end{bmatrix}\begin{bmatrix}\Delta x_i \\ \Delta y_i\end{bmatrix}. \quad (7)$$

## III. Solution of the Minimisation Problem

In order to solve the minimization problem, we use the classical Gauss-Newton algorithm with the Levenberg-Marquardt correction [5], [6]. This algorithm finds the vector $b$ which minimizes (5) in an iterative way. It approximates the objective function with the square of the norm of a linear function. Thus, at each iteration, the linear least-squares problem is solved

$$\min_{\delta b} ||\varepsilon'^{(k)} - \nabla\varepsilon'^{(k)}\cdot\delta b||_2 \quad (8)$$

where $\nabla\varepsilon'^{(k)}$ is the Jacobian matrix of first partial derivatives of $\varepsilon'$ with respect to $b$ and $\varepsilon'^{(k)}$ is $\varepsilon'$, both evaluated at $b^{(k)}$. A detailed description of the Levenberg-Marquardt algorithm can be found at [6]. In our case, the starting estimate for the centre coordinates and radius is obtained using the Taubin's approximation to the gradient-weighted algebraic circle fitting approach [5]. Then, the $\varphi_i$ values can be approximated as the angle between the x axis and the line joining the estimated circle centre and $\{x_i, y_i\}$.

Finally, to obtain the variance matrix of the centre coordinates and radius, an estimate of the variance matrix of the vector $b$ must be obtained. If $b = b' + \delta b$, where $\delta b \to 0$, then expanding $\varepsilon'^T\varepsilon'$ about the true value $b'$, we have $\nabla\varepsilon'^T\nabla\varepsilon'\delta b = -\nabla\varepsilon'^T\varepsilon'$. From this, we estimate the variance matrix of $\delta b$ and hence $b$

$$\begin{aligned} var(b) &= E[(b-b')(b-b')^T] \\ &= E[(\nabla\varepsilon'^T\varepsilon')^{-1}\nabla\varepsilon'^T\varepsilon'\varepsilon'^T\nabla\varepsilon'(\nabla\varepsilon'^T\nabla\varepsilon')^{-1}] \\ &= (\nabla\varepsilon'^T\nabla\varepsilon')^{-1}\nabla\varepsilon'^T E[\varepsilon'\varepsilon'^T]\nabla\varepsilon'(\nabla\varepsilon'^T\nabla\varepsilon')^{-1} \end{aligned} \quad (9)$$

being $E[\varepsilon'\varepsilon'^T] = s^2 I$, where $I$ is the identity matrix of order $2m$ and $s^2$ is the mean square error $(\varepsilon'^T\varepsilon'/(m-3))$, an unbiased estimator for $\sigma^2$ [4]. Substituting this into (8) gives $var(b) = s^2(\nabla\varepsilon'^T\nabla\varepsilon')^{-1}$. Rows and columns $2m+1$ to $2m+3$ of $var(b)$ give the variance and covariance of pairs of the parameter vector elements $(x_c, y_c, \rho)$.

## IV. Numerical Results and Applications to Mobile Robotics

To obtain an estimation of its pose, it is a common choice that a mobile robot carries a 2-D laser sensor whose data can be matched with the information contained in a map. Although the matching process between sensed data and stored one can be achieved using point-based algorithms, it is expected that algorithms based on parameterised geometric features will be more efficient. These feature-based matching algorithms transform the raw laser scans into geometric features. Among many geometric primitives, circle is very popular because it can be used to represent different items which are commonly found in structured and unstructured environments (columns, tree trunks,...) [2], [3].

In mobile robotics, although laser range readings are provided in the laser scanner's polar coordinate system, circle fitting approaches usually work in Cartesian coordinates. Therefore, they do not take advantage of the structure of the laser measurements. These approaches are also based on the assumption that the measurement errors are instances of independent and identically distributed random variables. However, range readings being processed in Cartesian coordinates $\{(x,y)_i\}$ are the result of a transformation of points from polar coordinates $\{(r,\phi)_i\}$, on which $r_i$ is the measured distance of an obstacle to the sensor rotating axis at direction $\phi_i$. This transformation adopts the nonlinear expression

$$x_i = r_i\cos\phi_i \quad y_i = r_i\sin\phi_i. \quad (10)$$

This makes errors in both Cartesian coordinates correlated [2]. Effectively, if the errors in range and bearing are assumed to be independent with zero mean and standard deviations $\sigma_r$ and $\sigma_\phi$, respectively, then the covariance matrix associated to a range reading $i$ in Cartesian coordinates can be approximated with a first-order Taylor expansion as

$$\begin{aligned} C_{xyi} &= J_r\begin{bmatrix}\sigma_\phi^2 & 0 \\ 0 & \sigma_r^2\end{bmatrix}J_r^T \\ &= \begin{bmatrix}\sigma_r^2 c^2 + r_i^2\sigma_\phi^2 s^2 & \sigma_r^2 sc - r_i^2\sigma_\phi^2 cs \\ \sigma_r^2 sc - r_i^2\sigma_\phi^2 cs & \sigma_r^2 s^2 + r_i^2\sigma_\phi^2 c^2\end{bmatrix} \end{aligned} \quad (11)$$

where $J_r$ is the Jacobian of the polar to Cartesian coordinates transformation, $c$ and $s$ are $\cos\phi_i$ and $\sin\phi_i$, respectively. It is shown the existence of nonzero terms associated to $p_i^2$, $q_i^2$ and $r_i$.

As it has been shown in previous Sections, the proposed method deals with the case in which the errors in both coordinates of the range readings are not independent. Besides, in order to provide an accurate feature estimation, it is also essential to represent uncertainties and to propagate them from single range reading measurements to all stages involved in the feature estimation process. In addition to estimate the circle

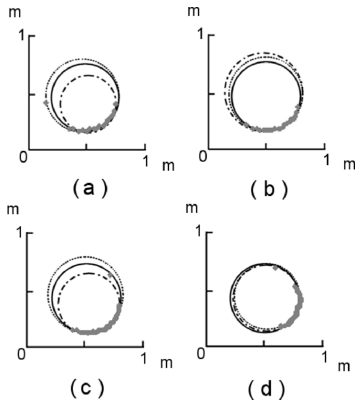


Fig. 2. (a)–(d) Fits to a set of range readings (grey points). Encoding is Zhang *et al.*'s approach [3]: dotted and Bailey's approach [7]: dashed; Proposed method: solid.

parameters, our method provides their associated variance matrix.

In order to demonstrate the performance of our method in a statistical setting, a set of artificial maps have been created using the Mapper3 software from Activmedia Robotics. Laser scans have been obtained from these maps using MobileSim, a software for mobile robots' simulation in any environment. This permits us to have the ground truth associated to the perceived maps, being possible to compare obtained and real circle parameters. In order to obtain realistic measures, simulated laser sensor provides range readings in polar coordinates which exhibit statistical errors on range ($\sigma_r = 5$ mm) and bearing ($\sigma_\rho = 0.1$ degrees). These errors are similar to the ones provided by a real SICK 200 Laser Measurement System [8]. From these, range readings in Cartesian coordinates can be obtained [2]. Each test scan represents an unique circular shape which can be partially occluded by other obstacles. A total of 100 scans were generated for testing. Tests have been conducted on a Pentium 4 1.7 GHz. As an example, Fig. 2 shows four different scans.

The proposed method has been compared to four similar approaches to test its performance. Thus, we have chosen for comparison the approaches developed by Zhang *et al.* [3] and Bailey [7]. These two approaches have been proposed for fitting a circle to a set of laser range readings in a mobile robot navigation framework. The Zhang *et al.*'s algorithm (ZA) minimizes the sum of squares of algebraic distances using a modified Gauss-Newton optimization approach. The starting guess for the parameter vector is obtained using the first three points. This does not give an accurate value and therefore, the algorithm needs more iterations to converge. Besides, this nonlinear least-squares problem could be efficiently solved by adopting an alternative parameterization [5]. In our approach, the selection of an initial guess is made using the Taubin's approximation to the gradient-weighted algebraic circle fitting approach [5]. This method will be also used for comparison (TA). On the other hand, the Bailey's proposal (BA) is a heuristic algorithm which estimate the circle parameters by finding the three range readings that make the triangle of maximum area. Finally, we have also chosen for comparison an alternative Levenberg-Marquardt approach proposed by Chernov and Lesort (CL) [5]. The Pratt's approximation to the gradient-weighted algebraic circle fitting method is used to select an initial guess of parameters. Fig. 2(a)–(d) show four laser scans associated to the same circular obstacle obtained from different poses. The detected circles obtained by the BA, ZA, and the proposed method are over-imposed on the range readings in Fig. 2(a)–(d). The results provided by the CL and TA have not been shown as they are very similar to the ones provided by the proposed approach. All methods provide good results, specially when the obstacle is observed from a close pose [Fig. 2(d)]. On the contrary, when range readings are sampled along a partially occluded circle, the Bailey's approach tends to return small circles which do not correctly fit the true circle. Table I shows the behaviour of each method in terms of computational workload and precision for the 100 scans according to the relative errors of the estimated centre coordinates and radius of each circular obstacle. These error measurements are defined as [3]

TABLE I
COMPARISON OF THE ERRORS $CE$ AND $RE$ (MM) AND AVERAGE PROCESSING TIME $t$ (MS). ERRORS MEAN AND VARIANCE OF THE 100 SCANS ARE PROVIDED

| | $E(CE)$ | $var(CE)$ | $E(RE)$ | $var(RE)$ | $t$ |
|---|---|---|---|---|---|
| BA [7] | 23.5 | 17.7 | 213.0 | 2050.1 | 5 |
| ZA [3] | 15.7 | 3.3 | 48.6 | 1217.5 | 13 |
| TA [5] | 12.5 | 4.5 | 42.7 | 1317.8 | 7 |
| CL [5] | 9.8 | 2.9 | 37.8 | 1018.0 | 8 |
| Proposed | 11.0 | 3.1 | 39.2 | 1281.2 | 10 |

$$CE = \frac{\sqrt{(x_{c,true} - x_c)^2 + (y_{c,true} - y_c)^2}}{\sqrt{x_{c,true}^2 + y_{c,true}^2}} \tag{12}$$

$$RE = \frac{|\rho_{true} - \rho|}{\rho_{true}} \tag{13}$$

where $(x_{c,true}, y_{c,true}, \rho_{true})$ are the true circle parameters and $(x_c, y_c, \rho)$ are the estimated ones. From Table I, it can be concluded that the proposed method works fine, although the Chernov and Lesort's approach provides the best results. Thus, it is more accurate than the Zhang *et al.*'s and Bailey's approaches. On the other hand, it is faster than the Zhang *et al.*'s proposal but slower than the rest of algorithms. Finally, it can be also noted that the proposed method provides best results than the Taubin's approach, which is used to obtain the initial guess in our algorithm.

## V. CONCLUSIONS

This paper proposes a circle fitting algorithm which works in the Cartesian coordinates space. Contrary to other approaches, this algorithm deals with the case in which the errors in both coordinates of the range readings are not independent. Then, it can be employed to fit a circle to a set of range readings when these are provided in Cartesian coordinates. Besides, the algorithm provides the variance matrix associated to the estimated circle parameters.

Finally, it must be noted that the parameterization of the problem requires to estimate a set of $m + 3$ parameters. This could result in increasing the number of local minima and reduce the accuracy of the approach. Although in our tests, the length of the minimum circular arc (over 40 degrees) limits the frequency of appearance of local minima, future work will be focused on a deeper analysis of this issue.

## ACKNOWLEDGMENT

The authors would like to thank the anonymous referees for their suggestions and insightful comments.

## REFERENCES

[1] D. Umbach and K. N. Jones, "A few methods for fitting circles to data," *IEEE Trans. Instrum. Meas.*, vol. 52, no. 6, pp. 1881–1885, Jun. 2003.
[2] A. Diosi and L. Kleeman, Uncertainty of Line Segments Extracted From Static Sick Pls Laser Scans, Tech. Rep. Dept. Elect. Comput. Syst. Eng., Monash Univ., Australia, 2003, MECSE-26-2003.
[3] S. Zhang, L. Xie, and M. D. Adams, "Feature extraction for outdoor mobile robot navigation based on a modified Gauss-Newton optimization approach," *Robot. Autonom. Syst.*, vol. 54, pp. 277–287, 2006.
[4] M. G. Cox and H. M. Jones, "An algorithm for least-squares circle fitting to data with specified uncertainty ellipses," *IMA J. Numer. Anal.*, vol. 9, pp. 285–298, 1989.
[5] N. Chernov and C. Lesort, "Least squares fitting of circles," *J. Math. Imag. Vis.*, vol. 23, pp. 239–251, 2005.
[6] C. Shakarji, "Least-squares fitting algorithms of the NIST algorithm testing system," *J. Res. Nat. Inst. Stand. Technol.*, vol. 103, pp. 633–641, 1998.
[7] T. Bailey, "Mobile robot localization and mapping in extensive outdoor environment," Ph.D. dissertation, Univ. Sydney, Sydney, Australia, 2002.
[8] P. Núñez, R. Vázquez-Martín, J. C. del Toro, A. Bandera, and F. Sandoval, "Feature extraction from laser scan data based on curvature estimation for mobile robotics," in *Proc. IEEE Int. Conf. Robotics and Automation*, 2006, pp. 1167–1172.